Remembering John Napier and His Logarithms

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Preface

My decision to first read and study The Construction of the Wonderful Canon of Logarithms was not motivated by an interest in logarithms. I was trying to learn more about the origin of the natural exponential function. I had known Leonhard Euler (1707 – 1783) probably did the most to engender wide spread understanding and acceptance of modern exponential functions (including the natural exponential function), but John Napier had over one hundred years earlier written a couple of papers creating something he called logarithms, which through hindsight, we have learned have a very intimate relationship with the natural exponential function. I wanted to understand this relationship with the hope that it would help me understand the context in which the natural exponential function was born.

I also discovered something unexpected. Napier’s Canon is a marvelous example of how engineering problems are solved in practice. Often times engineers are prohibited from directly applying textbook solutions because of the complexity of the problem. They, instead, make simplifying assumptions and estimate quantities of interest. Napier faced a similar situation. He defined logarithms, yet he could not directly compute them. So, he computed their estimates. His ability to do so, I believe, shows the real ingenuity and inspiration of John Napier.

Throughout this paper, I have tried to preserve most of Napier’s original proofs and arguments, updating the language and mathematics where necessary. At the end, I added a section hopefully answering some nagging questions a modern reader might have about a 400 year old paper. Such as, why did Napier choose the word logarithm and how do Napierian logarithms relate to ordinary modern logarithms? I enjoy placing mathematics into historical context and I hope the reader gains a new understanding and appreciation of logarithms which goes beyond what is usually gleaned from modern texts.

1 Introduction

John Napier (1550 – 1617) was a laird of the Merichston estate near Edinburgh, Scotland. He was not employed as a professional mathematician, although he is now most remembered as one of the inventors of logarithms. From what is known about his life, Napier spent a considerable amount of time studying mathematics searching for easier and more efficient ways of multiplying numbers [1,2]. During the late sixteenth and early seventeenth centuries, multiplication as well as, division and the extraction of roots were in general slow and tedious calculations. The invention of logarithms almost certainly, came as a long awaited relief to the labor of these calculations.
Napier

The sum of Napier’s work on logarithms is found in two treatises, The Description of the Wonderful Canon of Logarithms and The Construction of the Wonderful Canon of Logarithms. The Description was published in 1614 and the Construction, although written before the Description, was only published posthumously by his son Robert Napier in 1619. This paper focuses on the ideas and arguments presented by Napier in the Construction.

Napier devised an ingenious mathematical tool without the advantages of modern mathematics. Differential and integral calculus had not been invented nor had exponential notation (terms like base and exponent were not routinely used until much later). In fact, most of our contemporary mathematical language did not exist, so Napier could not even express his thoughts as we would today. Consequently, he initially described logarithms through geometry and not as the inverse of the exponential function. It was in Napier’s lifetime, decimal notation began to be widely accepted and Johann Kepler (1571 – 1630) derived his laws of planetary motion.

Similar to modern mathematical texts, the Construction, is written in a more or less axiomatic format. Napier begins with a few basic definitions and then progressively builds on them. The logarithmic function he describes is not the natural logarithmic function known today, but forms the basis and very essence of modern logarithms. As the name implies, the bulk of the Construction explains how to tabulate values for this function.

Napier did not develop an explicit mathematical expression for the logarithm of a number. He estimated them by finding a number whose logarithm possesses upper and lower bounds that differ by an “insignificant amount”. He then reasoned the average of these bounds was a good estimate of the actual logarithm. Through these types of estimates and by taking advantage of certain properties of logarithms, Napier built his entire Table.

2 Arithmetic and Geometric Progressions

The first portion of the Construction is introductory, but nevertheless, presents some ideas which are seen throughout the treatise and which are crucial to understanding central concepts. Arithmetic and geometric progressions (sequences) are two such topics worth highlighting. An arithmetical progression “proceeds by equal intervals” (Article 2) such that succeeding terms differ by a constant. A geometrical progression advances by “unequal but proportionally increasing or decreasing intervals”. That is, the ratio of succeeding terms is constant. As examples, Napier offers

Arithmetical progressions: 1, 2, 3, 4, 5, 6, 7, \ldots
2, 4, 6, 8, 10, 12, 14, 16, \ldots

Geometrical progressions: 1, 2, 4, 8, 16, 32, 64, \ldots
243, 81, 27, 9, 3, 1, \ldots

Despite the fact that the terms in these examples are integers, Napier required them to be real numbers in his definition. (This fact is important to remember but easily forgotten when one delves into the construction of the Table.) If Napier had modern notation, he might have described arithmetic and geometric motion by the functions

\text{Arithmetic motion: } f(t) = ct + b
\text{Geometric motion: } f(t) = ca^t

where in both cases \( f(\cdot) \) is a real valued function of a real variable, and \( a, b, c \in \mathbb{R} \).
3 Definition

Today, the natural logarithmic function is usually defined as the inverse of the natural exponential function (e.g. [3]), or through the integral equation

\[ \ln x = \int_1^x \frac{1}{t} \, dt, \quad x > 0 \]

(e.g. [4]). Napier offered a more qualitative definition. Consider two points \( \alpha, \beta \) moving along the lines shown in Figure 1. Let \( \alpha \) move arithmetically from left to right along the first line such that in equal time increments \( T \) it moves equal distances. In other words, let \( \alpha \) travel with constant velocity. Set the length of the second line equal to \( 10^7 \) and let \( \beta \) travel geometrically from left to right so that the distance travelled in each time increment equals a constant fractional proportion of the remaining length. (Napier’s choice of \( 10^7 \) is discussed below.) Since these lengths decrease, the velocity of \( \beta \) decreases in time.

Set the magnitude of the velocity of \( \alpha \) equal to \( 10^7 \) and require the velocity of \( \beta \) at any moment in time be equal (in magnitude) to the remaining distance at the same time. Thus at time 0, the velocity of \( \beta \) equals \( 10^7 \), at time \( T \) the velocity would equal \( 10^7 - x_{\beta_1} \), and at time \( 2T \), it would equal \( 10^7 - x_{\beta_2} \). “Hence, whatever be the proportion of the distances \([10^7, 10^7 - x_{\beta_1}, 10^7 - x_{\beta_2}, 10^7 - x_{\beta_3}],\) etc. to each other, that of the velocities of \([\beta] \) at the points \([0, x_{\beta_1}, x_{\beta_2}, x_{\beta_3}],\) etc. to one another, will be the same.” (Article 25)

Finally, let \( \beta \) move geometrically as described above from 0 to \( x_{\beta_k} \) in time \( kT \) and let \( \alpha \) move arithmetically (with constant velocity equal to \( 10^7 \)) for the same amount of time from 0 to \( x_{\alpha_k} \). The distance \( x_{\alpha_k} \) is called the logarithm of the distance \( 10^7 - x_{\beta_k} \). (Article 26)

To the modern reader this definition may seem imprecise, nevertheless, it explicitly expresses the relationship between a number and its logarithm. In particular, setting the velocity of \( \beta \) equal to the lengths of the remaining distances forced the base of the exponential function describing the motion of \( \beta \) to be \( e^{-1} \). (Refer to section 5 for a more complete explanation.)

Thus, knowing this base, Napier’s definition can easily be restated in modern terminology. Let the functions

\[ x_{\alpha}(t) = 10^7 t \quad (1) \]
\[ x_{\beta}(t) = 10^7 - 10^7 e^{-t} \quad (2) \]
describe the motion of \( \alpha \) and \( \beta \) respectively. Then the function

\[ z_\beta(t) = 10^7 e^{-t} \]  

(3)

describes the remaining distance from \( \beta \) to the end of the second line as a function of time.

Thus for a given time \( kT \), the Napierian logarithm of \( z(kT) \) is defined as

\[ \text{Nap} \log (z(kT)) = x_\alpha(kT). \]  

(4)

Napier chose to tabulate the logarithms of sines of angles because many intensive computational problems of the day involved trigonometry. Note that in the sixteenth century the sine of an angle was not normalized to a circle of radius one. Instead, it depended on the radius of the circle of interest. Specifically, the sine of an angle \( \theta \) was defined as the half chord \( AB \) of a circle of radius \( r \). Refer to Figure 3. The length of the second line represents the radius of Napier’s circle.

He chose \( 10^7 \), in particular, because he wanted to guarantee accuracy and ease of computation. (Article 3) Multiplying everything by \( 10^7 \) produced large integral numbers and achieved this goal.

Unfortunately, requiring the length of the second line to equal \( 10^7 \) instead of one, convolutes the computational advantages normally seen in modern logarithms. Napier himself was aware of the improvements which would be gained by rescaling and he, in fact, proposed a new and “better kind” of logarithm in the appendix of the Construction.

4 Construction of the Table

Besides reflecting his approach, the arrangement of Napier’s Table and his method of constructing it reflect the challenge of computing logarithms. As said before, his definition did not provide an explicit expression. Calculating the logarithm of a number wasn’t simply a matter of plugging numbers into an equation. Napier cleverly built his Table in a specific way so that he could take advantage of certain properties and arrive at estimates for his logarithms. He began by creating a decreasing geometric sequence which was easy to compute. Starting with the number \( a_0 = 10^7 \), he subtracted one one millionth from it to obtain the second number in the sequence. That is, he subtracted 1 from \( 10^7 \) to obtain 9999999.00000. Similarly, he subtracted one one millionth of 9999999.00000 to obtain the third number, i.e. \( a_2 = a_1 - a_1(.0000001) = 9999998.0000001 \). In general, \( a_n = a_{n-1} - a_{n-1}(.0000001) \). Napier continued the sequence for 100 iterations and called it the First Table (Table 1).
## Table 1: Napier’s First and Second Tables of decreasing geometric progressions. The first proceeds with a proportion of one one millionth and the second with one one hundred thousandth.

<table>
<thead>
<tr>
<th>1st Column</th>
<th>2nd Column</th>
<th>3rd Column</th>
<th>...</th>
<th>69th Column</th>
</tr>
</thead>
<tbody>
<tr>
<td>10000000.000000</td>
<td>9900000.000000</td>
<td>9801000.000000</td>
<td>...</td>
<td>5048858.8878707</td>
</tr>
<tr>
<td>9995000.000000</td>
<td>9895050.000000</td>
<td>9796099.500000</td>
<td>...</td>
<td>5046334.4584268</td>
</tr>
<tr>
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<td>9890102.475000</td>
<td>9791201.450250</td>
<td>...</td>
<td>5043811.2911976</td>
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<td>9885157.4237625</td>
<td>9786305.8495249</td>
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<td>5041289.3855520</td>
</tr>
<tr>
<td>9980014.995000</td>
<td>9880214.8450506</td>
<td>9781412.6966001</td>
<td>...</td>
<td>5038768.7408592</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>9900473.5780233</td>
<td>9801468.8422431</td>
<td>9703454.1538206</td>
<td>...</td>
<td>4998609.4018532</td>
</tr>
</tbody>
</table>

## Table 2: Napier’s Third Table. The proportion in each column is one two thousandth and one one hundredth in each row.

Next, he created a second decreasing geometric progression starting again with the number $10^7$ but using a different proportion. This time in order to make everything fit together, he looked at the first and last elements in the First Table and noted that their difference was roughly 100 which is one one hundred thousandth of $10^7$. He used this proportion to build his Second Table. It continued in the same manner as the First Table, but only for 50 iterations ending with the number 9995001.224804 (Table 1). He did not use the exact proportion existing between the first and last elements of the First Table for the very simple reason that using a round number like 100 made the computations easier.

Building on these two tables, Napier created a larger Third Table. See Table 2. This table had 21 rows and 69 columns and each row and column was a decreasing geometric sequence. The proportion in each column was one two thousandth; in each row one one hundredth. The first element in the first column was $10^7$ like before and the last element in the 69th column was 9995001.222927. Napier most likely made a computational error when constructing the Second Table. This error impacted later calculations, the effects of which are seen in Napier’s final logarithm table. [1]

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1This number mistakenly appeared in the Construction as 9995001.222927. Napier most likely made a computational error when constructing the Second Table. This error impacted later calculations, the effects of which are seen in Napier’s final logarithm table. [1]
Napier

\[ \alpha \quad \beta \]
\[ D \quad A \quad C \quad B \]
\[ E \quad F \]

Figure 3: The logarithm of the line segment \( CB \) has an upper bound equal to the length of \( DA \) and a lower bound equal to the length of \( AC \).

Napier began with these decreasing geometric progressions because his goal was to build a table which listed the logarithms of sines between zero and ninety degrees. This meant, given the archaic definition of the sine of an angle (and a radius of \( 10^7 \)), he was ultimately searching for the logarithms of 0 through \( 10^7 \). He did not tabulate progressions with a proportion of \( e^{-1} \) since he could not directly compute the terms. He instead created progressions which were easy to compute and which yielded estimates of their logarithms.

It follows directly from Napier’s definition that the logarithm of \( 10^7 \) is zero since the distance \( \alpha \) has traveled at time 0 is 0. Napier estimated the logarithm of 9999999.0000000 by introducing the following bounds.

Consider the abstraction of Figure 1 shown in Figure 3. Denote the endpoints of the line along which \( \beta \) travels as \( A \) and \( B \) and fix a point \( C \) arbitrarily between them. Extend this line to the left to the point \( D \) such that the length of the line segment \( DB \) is in the same proportion to \( AB \) as \( AB \) is to \( CB \), i.e. let

\[ \frac{DB}{AB} = \frac{AB}{CB}. \]

Allow \( \beta \) to move geometrically to the right beginning at \( D \) such that it moves from \( D \) to \( A \) and from \( A \) to \( C \) in equal times (the speed of \( \beta \) at \( D \) is equal to the length of the line segment \( DB \) which is greater than \( 10^7 \)). Allow \( \alpha \) to move arithmetically from \( E \) to \( F \) in the same amount of time.

The logarithm of the length of the segment \( CB \) is, by definition, the length of \( EF \). From the figure we also see that the logarithm of \( CB \) is greater than the length of the segment \( AC \) because \( \alpha \) moves at a constant speed as it travels from \( E \) to \( F \) while \( \beta \) continuously slows down as it moves from \( A \) to \( C \). Thus, the length of \( AC \) is a lower bound of the length of \( EF \). Similarly, the length of \( DA \) is an upper bound since \( \beta \) moves at higher speeds between \( D \) and \( A \) than \( \alpha \) does between \( E \) and \( F \).

Expressions for these bounds are easily found. Note that the length of \( AC \) equals the length of \( AB \) minus \( CB \), that is \( 10^7 \) minus the number with which you are taking the logarithm. An expression for the upper bound is found by noticing that the segment \( DA \) is in the same proportion to \( AB \) as \( AC \) is to \( CB \). This yields

\[ DA = \frac{(AB)(AC)}{CB}. \]

With these bounds Napier returned to the problem of computing the the logarithm of 9999999.0000000. If the length of \( CB \) equals 9999999.0000000, the lengths of \( AC \) and \( DA \) equal 1.0000000 and 1.0000001, respectively. Napier reasoned that these bounds differed “insensibly” (Article 31) and that any number between the bounds sufficed as an estimate of the
logarithm. He decided to take the average of the bounds and set the logarithm of 999999.0000000 equal to 1.0000005.

If Napier had started with a different proportion in his progressions, his bounds could have very easily yielded an inaccurate estimate. If, for example, he used one tenth instead of one ten millionth in his First Table, he would have computed the bounds of 9000000.000000 as 1000000.0 and 11111111.11!

It is evident from Napier’s definition that as the remaining lengths decrease geometrically, their logarithms increase arithmetically. For a given time interval $T$, $\beta$ will move from 0 to $x_{\beta1}$ (Figure 4) and by definition, the logarithm of $10^7 - x_{\beta1}$ is $x_{\alpha1}$. At time $2T$, the remaining length becomes $10^7 - x_{\beta2}$ and the logarithm of this distance is simply twice the logarithm of $10^7 - x_{\beta1}$ since $x_{\alpha2}$ is twice $x_{\alpha1}$. Likewise, the logarithm of $10^7 - x_{\beta3}$ equals three times $x_{\alpha1}$.

Using this logic, Napier took his estimate for the logarithm of 999999.0000000 and doubled, tripled, quadrupled, etc. it and computed all the logarithms of the First Table (Table 3).

<table>
<thead>
<tr>
<th>Number</th>
<th>Logarithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>10000000.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>9999999.0000000</td>
<td>1.0000000</td>
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<tr>
<td>9999998.0000001</td>
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</tr>
<tr>
<td>9999990.0000045</td>
<td>10.0000005</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>9999900.0004950</td>
<td>100.0000050</td>
</tr>
</tbody>
</table>

Table 3: Completed First Table.
Napier

Napier now tackled the Second Table. The first value is $10^7$ which we already know has a logarithm equal to zero. To find logarithm of 9999900.000000, Napier proceeded essentially as he did before seeking an upper and a lower bound which differed by an insensible amount. This time, though, the process becomes more complicated. It involves three more properties.

1. The difference of the logarithm of a given number and the logarithm of $10^7$ is just the logarithm of the given number (Article 34).

   **Proof:** This is self-evident since the logarithm of $10^7$ is zero. □

2. The logarithms of similarly proportioned numbers are equidifferent (Article 36).

   **Proof:** This property follows directly from Napier's definition since the point $\beta$ will travel for equal time increments between any two numbers that are similarly proportioned. *Example:* From Table 3, the proportion of the numbers 9999993.0000000001 and 9999991.0000003600 is $1 : 5000000$, and the difference of their logarithms is 2.0. The numbers 9999992.0000028 and 9999990.0000045 are similarly proportioned and the difference of their logarithms $(10.0000005 - 8.0000004)$ is also 2.0. □

3. The difference of two logarithms is bounded (Article 39).

   **Proof:** Refer to Figure 5. Let the line segment $AB$ have length $10^7$ and let the segments $CB$ and $DB$ represent the numbers whose logarithms are of interest. Define the points $E$ and $F$ such that

   \[
   \frac{EA}{AB} = \frac{CD}{DB}, \tag{5}
   \]

   and

   \[
   \frac{AF}{AB} = \frac{CD}{CB}. \tag{6}
   \]

   Solving equation 5 for $EA$, we see that $EA = (AB)(CD)/DB$ and thus

   \[
   \frac{EB}{AB} = \frac{(AB)(CD)}{DB} + \frac{AB}{AB} = \frac{AB(CD + 1)}{AB} = \frac{CD + DB}{DB} = \frac{CB}{DB}.
   \]

   Therefore by the second property, the logarithms of $CB$ and $DB$ differ by the same amount as the logarithms of $AB$ and $FB$ since they are similarly proportional. By the first property, this difference is simply equal to the logarithm of $FB$, and as before, is bounded by $EA$ and $AF$. Therefore, Napier concluded that the difference of two logarithms is bounded.

   \[
   \frac{(AB)(CD)}{CB} = AF < \text{Nap log } (CB) - \text{Nap log } (DB) < EA = \frac{(AB)(CD)}{DB} \tag{7}
   \]

   This last property is the key to finding the bounds of the logarithm of 9999900.000000 and to determining all of the logarithms of the Second Table. Specifically, Napier found the
number closest to 9999900.0000000 in the First Table (9999900.0004950) and noted its logarithm (100.0000000) and bounds (100.00000100 and 100.0000000). Using property 3, he determined that .0004950 should be added to the bounds of 9999900.0004950 to yield the bounds of 9999900.0000000. The result being 100.0005050 and 100.0004950. Finally, as before, Napier reasoned these bounds differed by an insensible amount so either bound or any number between them could be taken to be the true logarithm of 9999900.0000000. Thus, Napier completed the Second Table by repeating what he had done for the First. He doubled, tripled, quadrupled, etc. the logarithm of 9999900.0000000 as he stepped through the progression. See Table 4.

<table>
<thead>
<tr>
<th>Number</th>
<th>Logarithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>10000000.000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>9999900.000000</td>
<td>100.0005000</td>
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</tr>
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<td>900.0045000</td>
</tr>
<tr>
<td></td>
<td>⋮</td>
</tr>
<tr>
<td>9995001.2248040</td>
<td>5000.0250000</td>
</tr>
</tbody>
</table>

Table 4: Completed Second Table.

Napier was now ready to find the logarithms of the Third Table. The problem boiled down to finding the logarithms of two numbers. If he could find the logarithm of 9995000.0000000 (first element, first column), he could calculate all of the logarithms in the first column since they are in a particular geometric progression. Likewise, if he could calculate the logarithm of 9900000.0000000 (first element, second column), he could fill in all of the remaining columns since all of the these elements are geometrically related. Using the same method as before, Napier determined the upper and lower bounds for the logarithm of 9995000.0000000. The closest number in the Second Table to 9995000.0000000 is 9995001.224804. The bounds for this logarithm as calculated before are 5000.02525 and 5000.02475. Seeking the bounds for the difference of the logarithms results in 1.2254167 and 1.2254165 for the upper and lower bound respectively. Adding these bounds to 5000.02525 and 5000.02475 yields the bounds for the logarithm of 9995000.0000000. These are 5001.2506667 and 5001.2501665. The true logarithm of 9995000.0000000 was taken to be the average of these bounds or 5001.2504166. Following the same process, Napier determined the logarithm of 9900000.0000000 to be 100503.35853072.
Taking 5001.2504166 to be the logarithm of 9995000.0000000, Napier once again doubled, tripled, quadrupled, etc. it to determine the logarithms for all of the numbers in the first column of the Third Table. Then starting again with 5001.2504166, he successively added 100503.3585307 to it to obtain all of the logarithms of the first row. To fill in the remainder of the Table, one can either take the first element in each column and successively add to it 5001.2504166 to obtain the logarithms for each column, or take the first element in each row and successively add to it 100503.3585307 to obtain the logarithms for each row.

Having found all of the logarithms in the Third Table, Napier instructed the reader to arrange it as seen in Table 5 so that in his own words, it “may be made complete and perfect.” (Article 47) From this point forward, Napier referred to the Third Table as The Radical Table and stated that it be used to construct the Logarithmic Table of sines. The only work left undone was to determine how to calculate the logarithms of all the numbers in between the numbers tabulated in the Radical Table and to calculate the logarithms from zero to 5000000. Fortunately, Napier already had the answers. For any number greater than 9996700, the logarithm can be determined by simply computing its lower bound (subtract given number from $10^7$) as was done in estimating the logarithms of the First Table. To find the logarithm of any number embraced within the Radical Table, the bounds should be computed as was done in estimating the logarithms of the Second and Third Tables. To find the logarithms of numbers less than 5000000, Napier further exploited the property that the logarithms of similarly proportioned numbers are equidifferent. Given a number, one can multiple it by any convenient proportional factor and obtain a number which will lie within the limits of the Radical Table. This number is, of course, in proportion to the original number and its logarithm will differ from the logarithm of the original number by an amount dependent upon the proportional factor. The logarithm of the proportional number can be determined as before and all one has to do to obtain the logarithm of the original number is to add to it the difference of the logarithms.

Napier concluded by describing how his Logarithmic Table should be arranged. “Prepare forty-five pages” (Article 59) each with seven columns. Each page was devoted to two degrees. The first and the last columns each listed every minute within the two degrees in such a way that they were complements of each other. Wherefore, the first column began with 0 degrees, 0 minutes and ended with 0 degrees, 60 minutes, and the last column began with 89 degrees, 60 minutes and ended with 89 degrees, 0 minutes. In the second and sixth columns the corresponding sines were listed next to the angle with which it was associated and likewise, the logarithms of the sines were tabulated in the third and fifth columns. Lastly, the difference of the two logarithms from each row appeared in the fourth column. This was Napier’s Logarithmic Table, the culmination of over twenty years work and a milestone in the history of mathematics.

### 5 Remaining Questions

**Why Napier’s logarithms are essentially to the base $e^{-1}$?**

Napier required the velocity of $\beta$ to be the same as the remaining distances, hence he necessarily, if unknowingly, specified the base of its geometric motion. To see this, suppose

$$x_\beta(t) = 10^7 - 10^7(1 - p)^t$$  \hspace{1cm} (7)

describes the motion of $\beta$, where $p$ is an arbitrary proportional factor.
Thus, 
\[ z_\beta(t) = 10^7 (1 - p)^t \] (8)
describes the lengths of the remaining distances.

The velocity of \( \beta \) is the derivative of equation 7
\[ v_\beta(t) = \frac{d}{dt} \left[ 10^7 - 10^7(1 - p)^t \right] = -10^7(1 - p)^t \ln (1 - p). \] (9)

According to the definition, equations 8 and 9 must be equal. But equality holds only if 
\(- \ln (1 - p) = 1\), or if \(1 - p = 1/e\). Therefore, by definition, the base of the exponential function describing the motion of \( \beta \) is \(e\).

This fact has direct bearing on determining the base of Napier’s logarithms. If the total length of the second line equaled 1, Napier’s logarithms would exactly be to the base \(e^{-1}\). However, since Napier choose \(10^7\), he multiplied his numbers and logarithms by that amount. For example, the logarithm of 981277.6670907 as listed in the Radical Table is 190525.8660295. But the (modern) logarithm to the base \(e^{-1}\) of 190525.8660295/10^7 is 0.01905258660379. Multiplying this result by \(10^7\) yields an answer which agrees with Napier’s tabulated value up to eleven significant figures! The majority of the small error derives from the method of estimation for the logarithm of 9999999.

If the factor \(10^7\) is strictly taken into account, it can be argued that the true base is \(e^{-10^7}\) (shown below), however it is my opinion, that describing Napier’s logarithms as having base \(e^{-1}\) best preserves their true nature.

Relation between Napierian and modern logarithms

Solve equation 3 for \(kT\) to obtain
\[ kT = -\ln \left( \frac{z_\beta(kT)}{10^7} \right). \]

Substituting this result into equation 1 yields
\[ x_\alpha(kT) = -10^7 \ln \left( \frac{z_\beta(kT)}{10^7} \right). \]

Thus equation 4 becomes
\[ \text{Nap} \log z_\beta(kT) = -10^7 \ln \left( \frac{z_\beta(kT)}{10^7} \right) = -10^7 \ln z_\beta(kT) + 10^7 \ln 10^7. \]

Changing the base of the natural logarithm we can rewrite \(-10^7 \ln z_\beta(kT)\) as \(\log_b z_\beta(kT)\) where \(b = e^{-1/10^7}\). Therefore, Napierian logarithms can also be thought of as logarithms to the base \(b\) shifted by \(10^7 \ln 10^7\).

\[ \text{Nap} \log z_\beta(kT) = \log_b z_\beta(kT) + 10^7 \ln 10^7 \]

Why the word logarithm?

Napier coined the Latin word logarithmus which derives its meaning from two Greek words: logos meaning a principle relationship between numbers or ratio, and arithmos meaning number. Napier did not explain his view of the literal meaning of logarithmus, but seems appropriate since the concept of proportion is central to the idea of logarithms.
Napier

Utility

Modern logarithms derive most of their computational utility from the following three properties:

\[
\log_a(xy) = \log_a x + \log_a y \\
\log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y \\
\log_a(x^n) = n \log_a x
\]

Since Napier used \(10^7\) instead of 1, his logarithms do not strictly possess the first and third properties. However, if this factor is properly taken into account, his logarithms behave and can be used like modern logarithms. Certainly, Napier recognized the advantages early on, for he writes, “by it [his logarithmic table] all multiplications, divisions, and the more difficult extraction of roots are avoided.” (Article 1)

6 Conclusion

As a closing note, it should be remembered that Napier did not exist in a vacuum. The full development of logarithms sprang from the efforts of many individuals. Without Burgi, Briggs, Vlacq and many others logarithms would not have reached the mathematical maturity nor the usefulness they have. Napier himself welcomed collaboration and encouraged further development. For he said, “Nothing is perfect at birth.”

7 Acknowledgement

I thank Anthony F. Lexa, my father, for his comments and encouragement.

References


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Table 5: Radical Table